Analysis on Singularity of First-order Linear Differential Equations

Ling Hao

Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China

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Abstract: First-order linear differential equations are an important class of ordinary differential equations. They are the simplest differential equations with general solutions, generally can be expressed as $\frac{dy}{dx} = p(x)y + q(x)$. The complex geometry method is used to study the properties of the singularity of the first-order linear differential equation on the complex plane, prove that the singularity can only be on the x-axis, and calculate the eigenvalues of the singularity. At the same time, we proved that under sufficiently general conditions, the eigenvalues of the singularities are equal to the residues of $p(x)$. At this time, the eigenvalues of all the singularities of the equation satisfy the same overall properties as the residues of $p(x)$.

1. Introduction

The general form of the first-order linear ordinary differential equation is shown.

$$\frac{dy}{dx} = p(x)y + q(x).$$

Wherein, $p(x), q(x)$ is a continuous function on the interval.

Darboux first studied differential equations from the perspective of complex geometry in Ref.[1]. In fact, we can study differential equations from the perspectives of vector fields and foliage, see Refs.[2-7]. From a geometric point of view, the first-order linear ordinary differential equation can also be expressed as Equation (1).

$$\frac{dy}{dx} = \frac{p_1(x)}{p_2(x)}y + \frac{q_1(x)}{q_2(x)}. \quad (1)$$

Among them, $p_1(x), p_2(x)$ and $q_1(x), q_2(x)$ are polynomial functions that are relatively prime in the complex number domain. Rewrite the equation to get Equation (2).

$$\left(p_1(x)q_2(x)y + q_1(x)p_2(x)\right)dx - p_2(x)q_2(x)dy = 0. \quad (2)$$

Record $\omega = \left(p_1(x)q_2(x)y + q_1(x)p_2(x)\right)dx - p_2(x)q_2(x)dy$ as a 1-form, and the original differential equation can be recorded as $\omega=0$. Brunella pointed out in the Ref.[2] that the singularity of this differential equation is determined by the common zero of the polynomial $p_1(x)q_2(x)y + q_1(x)p_2(x)$ and $p_2(x)q_2(x)$.

From the perspective of the singularity of the differential equation, this paper analyzed the relationship between the singularity and the coefficient function of the differential equation step by step, and finally obtained the overall properties of the singularity eigenvalues of the first-order linear differential equation in a fully general situation.

Note: If the polynomial $p_1(x)q_2(x)y + q_1(x)p_2(x)$ and $p_2(x)q_2(x)$ are non-coprime, the common factor can be eliminated in Equation (2) to ensure that $p_1(x)q_2(x)y + q_1(x)p_2(x)$ and $p_2(x)q_2(x)$ are co-prime. The influence of this common factor on the singularity only exists in the degenerate singularity, see Ref.[8]. For the convenience of narration, we directly assume

$p_1(x)q_2(x)y+q_1(x)p_2(x)$ and $p_2(x)q_2(x)$ are co-prime.

2. Main conclusions and proofs

First of all, consider the zero point of $p_2(x)q_2(x)$, the zero point of the polynomial can be divided into two parts, denoted as $\text{Zeros}\{p_2(x)\}$ and $\text{Zeros}\{q_2(x)\}$, ; y can take any value.

Lemma 1 $p_2(x), q_2(x)$ are co-prime.

Proof: The common factor of $p_2(x)$ and $q_2(x)$ is also that of $p_1(x)q_2(x)y+q_1(x)p_2(x)$ and $p_2(x)q_2(x)$, so $p_2(x)$ and $q_2(x)$ are co-prime.

Theorem 1 is same to the differential equation of (2), the set of all singularities is $\{(x,0)\,|\,x\in \text{Zeros}\{p_2(x)\}\}$.

Proof: Now, consider the zero point of $p_1(x)q_2(x)y+q_1(x)p_2(x)$. Suppose the greatest common factor of $p_1(x), q_1(x)$ is $r(x)$, $p_1(x)=p_1'(x)r(x), q_1(x)=q_1'(x)r(x)$, . Then the zero point of $p_1(x)q_2(x)y+q_1(x)p_2(x)$ can only be the following two cases.

1) The zero point of $r(x)$. Obviously $\text{Zeros}\{r(x)\}$ is the zero point of $p_1(x)q_2(x)y+q_1(x)p_2(x)$.

2) $p_1(x)q_2(x)\neq 0$, $y=-\dfrac{q_1(x)p_2(x)}{p_1(x)q_2(x)}=-\dfrac{q_1'(x)p_2(x)}{p_1'(x)q_2(x)}$.

Case 1. When $r(x)$ is the greatest common factor of $p_1(x), q_1(x)$, $r(x)$ and $p_2(x)q_2(x)$ are co-prime, and the zero point of $r(x)$ is not the common zero point of $p_1(x)q_2(x)y+q_1(x)p_2(x)$ and $p_2(x)q_2(x)$.

Case 2. $q_2(x)\neq 0$, . In order to meet the zero point of $p_2(x)q_2(x)$, only $p_2(x)=0$. ; $y=-\dfrac{q_1'(x)p_2(x)}{p_1'(x)q_2(x)}=0$, , namely $\{(x,0)\,|\,x\in \text{Zeros}\{p_2(x)\}\}$ is the common zero point of $p_1(x)q_2(x)y+q_1(x)p_2(x)$ and $p_2(x)q_2(x)$.

Corollary 1 If $p_2(x)$ is a polynomial of m degree, the sum of multiplicity of singularities of a differential equation like (2) is m. When $p_2(x)$=1, that is there is no singularity of $p(x)$, the differential equation has no singularity.

Set $(x',0)$ as a singular point of the differential equation, let $x\to x-x'$, translate the singular point to the origin, and obtain the local equation of the original equation at the attachment of the singular point as Equation (3).

$$\left(p_1(x+x')q_2(x+x')y+q_1(x+x')p_2(x+x')\right)dx-p_2(x+x')q_2(x+x')dy=0. \quad (3)$$

$p_1(x')=a, p_2(x+x')=bx+o(x), q_1(x')=c, q_2(x')=d$, then Equation (3) can be transformed to Equation (4).

$$\left(ady+bcx+o(x,y)\right)dx-\left(bdx+o(x,y)\right)dy=0. \quad (4)$$

In Equation (4), $o(x,y)$ is the high-order infinitesimal of x, y. Therefore, the characteristic matrix of the singularity is followed.

$$\begin{pmatrix} bd & 0 \\ bc & ad \end{pmatrix}.$$

The eigenvalues of the matrix (singularity)is $\lambda_1=bd, \lambda_2=ad.$, obviously $\lambda_2\neq 0$, . The ratio of the eigenvalues is $\lambda = \frac{\lambda_1}{\lambda_2}$.(which can be called the eigenvalues of the singularity). If x' is a double or more zero point of $p_2(x)$, $b=0, \lambda_1 = 0$, $\lambda = 0$, the singularity is a saddle point. Otherwise $\lambda = \frac{b}{a}$ or $\frac{a}{b}$. About the eigenvalues (classification) of singularity of the differential equation can be found in Refs.[8-10].

Theorem 2: If the singularity $(x', 0)$ of a differential equation like (2) is not a saddle point, then $\lambda = \operatorname{Re}s\left[p(x), x'\right]$ or $\frac{1}{\operatorname{Re}s\left[p(x), x'\right]}$.

Proof: Note that x' is a single zero point of $p_2(x)$, and x' is not a zero point of $p_1(x)$. Therefore, from the residue calculation formula, we can get the following equation.

$$\operatorname{Re}s\left[p(x), x'\right] = \lim_{x \to x'} \frac{p_1(x)}{p_2'(x)} = \frac{a}{b}.$$

Corollary 1 Record $\lambda_p = \operatorname{Re}s\left[p(x), x_p\right]$ as the ratio of the eigenvalues corresponding to the singularity $p = (x_p, 0)$. When $p(x)$ is sufficiently general (that is, when there is no saddle point in the singularity of the differential equation like (2)), there is $\sum_p \lambda_p = -\operatorname{Re}s\left[p(x), \infty\right]$.

For example, suppose the differential equation is $\frac{dy}{dx} = \frac{1}{x^3+1} y$,, then the singularities of the equations are $(1, 0)$,, $(\frac{-1+\sqrt{3}i}{2}, 0)$, and $(\frac{-1-\sqrt{3}i}{2}, 0)$, and their eigenvalues are $\operatorname{Re}s\left[\frac{1}{x^3+1}, 1\right] = \frac{1}{3}$, $\operatorname{Re}s\left[\frac{1}{x^3+1}, \frac{-1+\sqrt{3}i}{2}\right] = -\frac{1-\sqrt{3}i}{6}$, , , and $\operatorname{Re}s\left[\frac{1}{x^3+1}, \frac{-1+\sqrt{3}i}{2}\right] = -\frac{1+\sqrt{3}i}{6}$. respectively. Since there is only one singularity, then there is $\sum_p \lambda_p = 0 = -\operatorname{Re}s\left[\frac{1}{x^3+1}, \infty\right]$..

3. Conclusion

In this paper, the overall properties satisfied by the eigenvalues of the singularities of the first-order differential equations are obtained in a sufficiently general situation, which lays a foundation for the study on the singularities of more complex differential equations. The next question worth studying is: when the singularity is a saddle point, what properties can the residue of $p(x)$ reflect the singularity?

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